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# A Bifurcation Theorem for Potential Operators\*

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A general bifurcation theorem for potential operators is proved. It describes the possible behavior of the set of solutions of an operator equation as a function of the eigenvalue parameter in a neighborhood of the bifurcation point. The theorem applies in particular to buckling problems in elasticity theory as well as to other fields in which the bifurcation problems have a variational formulation.

Let  $E$  be a real Banach space,  $\Omega$  a neighborhood of 0 in  $E$ , and  $C^k(\Omega, E)$  the set of  $k$  times continuously Fréchet differentiable maps from  $\Omega$  to  $E$ . Suppose  $L, H \in C(\Omega, E)$  with  $L$  linear and  $H(u) = o(\|u\|)$  at  $u = 0$ . Then for  $\lambda \in \mathbb{R}$ , the equation

$$Lu + H(u) = \lambda u \quad (0.1)$$

possesses the trivial family of solutions  $\{(\lambda, 0) \mid \lambda \in \mathbb{R}\}$ . Bifurcation theory is concerned in part with the existence of nontrivial solutions of (0.1) having  $\|u\|$  small. A point  $(\mu, 0) \in \mathbb{R} \times E$  is called a *bifurcation point* for (0.1) if every neighborhood of  $(\mu, 0)$  contains nontrivial solutions of (0.1). It is easily seen that a necessary condition for  $(\mu, 0)$  to be a bifurcation point is that  $\mu \in \sigma(L)$ , the spectrum of  $L$ . Simple examples show this condition is not sufficient. Diverse methods: analytical, topological, and variational, have been used in attempting to find sufficient conditions for bifurcation to occur. In this paper we are interested in variational methods. Most of the earlier work in this direction has dealt with the number of solutions of (0.1) near  $(\mu, 0)$  as a function of  $r = \|u\|$ . However, in many problems one is interested in how the number of solutions of (0.1) changes as  $\lambda$  varies. This question is the object of our study here.

Our main result is

**THEOREM 0.2.** *Let  $E$  be a real Hilbert space,  $\Omega$  a neighborhood of 0 in  $E$ , and  $f \in C^2(\Omega, \mathbb{R})$  with  $f'(u) = Lu + H(u)$ ,  $L$  being linear and  $H(u) = o(\|u\|)$  at*

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$u = 0$ . If  $\mu$  is an isolated eigenvalue of  $L$  of finite multiplicity, then  $(\mu, 0)$  is a bifurcation point for (0.1). Moreover, at least one of the following alternatives occurs:

- (i)  $(\mu, 0)$  is not an isolated solution of (0.1) in  $\{\mu\} \times E$ .
- (ii) There is a one-sided neighborhood,  $\Lambda$ , of  $\mu$  such that for all  $\lambda \in \Lambda \setminus \{\mu\}$ , (0.1) possesses at least two distinct nontrivial solutions.
- (iii) There is a neighborhood,  $I$ , of  $\mu$  such that for all  $\lambda \in I \setminus \{\mu\}$ , (0.1) possesses at least one nontrivial solution.

*Remark.* In the above theorem  $f'(u)$  denotes the Fréchet derivative of  $f$  at  $u$  and therefore should properly be interpreted as a linear map from  $E$  to  $\mathbb{R}$ , i.e., as an element of  $E'$ . However, since  $E$  is a Hilbert space, we can identify  $E'$  with  $E$  in (0.1). It is also easy to give examples where the various cases occur, e.g., (i) obtains when  $H \equiv 0$ .

The first results we know of for (0.1) using variational methods are due to Krasnoselski [1] who studied (0.1) under more stringent hypotheses on  $f$ . For his case  $L$  is compact (and symmetric) and therefore  $\sigma(L)$  consists only of real eigenvalues  $\mu$  of  $L$  of finite multiplicity. Krasnoselski used minimax arguments to prove that for every such eigenvalue,  $(\mu, 0)$  is a bifurcation point. In fact he showed for each  $r > 0$  and near 0, there is a solution  $(\lambda(r), u(r))$  of (0.1) having  $\|u(r)\| = r$  and in addition  $(\lambda(r), u(r)) \rightarrow (\mu, 0)$  as  $r \rightarrow 0$ . Extensions and improvements of Krasnoselski's work have been made by several people [2–10]. In particular Böhme [3] and Marino [4] have independently proved that under the hypotheses of Theorem 0.2, for each  $r > 0$  and near 0, there exist at least two distinct solutions of (0.1) having  $\|u\| = r$  and as in Krasnoselski's result. Recently McLeod and Turner [9] have weakened the requirement that  $f \in C^2$  to  $f \in C^1$  and  $f'$  Lipschitz continuous with a small Lipschitz constant.

The only other work we know of which studies the number of solutions of (0.1) as a function of  $\lambda$  in the variational case is due to Clark [10]. He assumed  $L$  compact,  $H(u) = T(u) + V(u)$  with  $T$  homogeneous of odd degree  $k \geq 3$ ,  $V(u) = o(\|u\|^k)$  at  $u = 0$  and odd, and  $0 \notin PV(u)$  (where  $P$  is the orthogonal projector of  $E$  onto  $N(L - \mu I)$ , the null space of  $L - \mu I$  if  $0 \neq u \in N(L - \mu I)$ ). Under these hypotheses Clark gives lower bounds on the number of solutions of (0.1) having  $\lambda > \mu$  (resp.  $\lambda < \mu$ ) and near  $\mu$  in terms of a topological measure of the size of the set where  $(T(u), u) \leq 0$  (resp.  $> 0$ ) on the unit sphere in  $N(L - \mu I)$ .

We will prove Theorem 0.2 in Section 1. More can be said when  $f$  is even. For this case in their framework Böhme and Marino showed for each small  $r > 0$  there are at least  $\dim N(L - \mu I)$  distinct pairs of solutions  $(\lambda, u)$  having  $\|u\| = r$ . We will prove an analog of this theorem in our setting in Section 2. Our result improves Clark's work but does not provide a completely satisfactory

answer to the question of the number of solutions of (0.1) as a function of  $\lambda$  for this case.

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## 1. PROOF OF THEOREM 0.2

The proof of Theorem 0.2 consists of several steps. First the so-called method of Lyapunov-Schmidt is used to reduce (0.1) to an equivalent finite-dimensional problem. Next this problem is cast as a variational problem. Then assuming alternative (i) of Theorem 0.2 does not occur, we have two cases corresponding to (ii) and (iii) of the theorem. Both are treated via minimax arguments. Case (ii) involves relatively standard arguments but (iii) is somewhat more subtle.

We begin with the Lyapunov-Schmidt reduction. Let  $X = N(L - \mu I)$ , the null space of  $L - \mu I$ . Suppose  $\dim X = n$ . Therefore we can identify  $X$  with  $\mathbb{R}^n$ . Let  $X^\perp$  denote the orthogonal complement of  $X$  in  $E$  and  $P, P^\perp$  denote the orthogonal projectors of  $E$  onto  $X, X^\perp$ , respectively. Then (0.1) is equivalent to the pair of equations

$$\mu v + PH(v + w) = \lambda v, \quad (1.1)$$

$$Lw + P^\perp H(v + w) = \lambda w, \quad (1.2)$$

where  $u = v + w$ ,  $v \in X$ ,  $w \in X^\perp$ . Define

$$F(\lambda, v, w) \equiv (L - \lambda I)w + P^\perp H(v + w).$$

Then  $F$  is continuously differentiable near  $(\mu, 0, 0) \in \mathbb{R} \times X \times X^\perp$ ,  $F(\mu, 0, 0) = 0$ , and the Fréchet derivative of  $F$  with respect to  $w$  at  $(\mu, 0, 0)$ ,  $F_w(\mu, 0, 0) = L - \mu I$  is an isomorphism of  $X^\perp$  onto  $X^\perp$ . Therefore by the implicit function theorem, there is a neighborhood  $\mathcal{O}$  of  $(\mu, 0) \in \mathbb{R} \times X$  and  $\varphi \in C^1(\mathcal{O}, X^\perp)$  such that the zeros of  $F$  near  $(\mu, 0, 0)$  are given by  $(\lambda, v, \varphi(\lambda, v))$  for  $(\lambda, v) \in \mathcal{O}$ . Thus solving (0.1) is equivalent to solving the finite-dimensional problem (1.1) with  $w = \varphi(\lambda, v)$  for  $(\lambda, v) \in \mathcal{O}$ .

Observe also that from (1.2) we have

$$\varphi(\lambda, v) = -(L - \mu I)^{-1} P^\perp H(v + \varphi(\lambda, v)) \quad (1.3)$$

where  $(L - \mu I)^{-1}$  is taken relative to  $X^\perp$ . Since  $H(u) = o(\|u\|)$  at  $u = 0$ , it follows that

$$\varphi(\lambda, v) = o(\|v\|) \quad \text{at } v = 0 \quad (1.4)$$

uniformly for  $\lambda$  near  $\mu$ .

Next we show that (1.1) is equivalent to a finite-dimensional variational problem. Let

$$\begin{aligned} g(\lambda, v) &\equiv f(v + \varphi) - (\lambda/2)(\|v\|^2 + \|\varphi\|^2) \\ &= \frac{1}{2}(\mu - \lambda)\|v\|^2 + \frac{1}{2}(L\varphi, \varphi) - (\lambda/2)\|\varphi\|^2 + h(v + \varphi) \end{aligned} \quad (1.5)$$

where  $\varphi = \varphi(\lambda, v)$ ,  $(\cdot, \cdot)$  denotes the inner product in  $E \times E$ ,  $h' = H$ , and  $h(0) = 0$ . We claim that for fixed  $\lambda$  near  $\mu$ , the critical points of  $g(\lambda, \cdot)$  near  $v = 0$  are solutions of (1.1). Indeed at a critical point of  $g(\lambda, \cdot)$  we have

$$(g_v(\lambda, v), \xi) = (f_v(v + \varphi), \xi) + (f_w(v + \varphi), \varphi_v(\lambda, v) \xi) - \lambda[(v, \xi) + (\varphi, \varphi_v(\lambda, v) \xi)] = 0 \quad (1.6)$$

for all  $\xi \in X$ . By (1.2),  $(f_w(v + \varphi) - \lambda\varphi, \psi) = 0$  for all  $\psi \in X^\perp$ . Therefore (1.6) reduces to

$$(g_v(\lambda, v), \xi) = (f_v(v + \varphi) - \lambda v, \xi) = 0 \quad (1.7)$$

for all  $\xi \in X$  which is equivalent to (1.1) with  $w = \varphi(\lambda, v)$ .

The above remarks demonstrate that to solve (0.1), it suffices to find critical points of  $g(\lambda, \cdot)$  near  $v = 0$  where  $\lambda$  is fixed and near  $\mu$ . Clearly  $v = 0$  is a critical point of  $g(\lambda, \cdot)$  for each such  $\lambda$ . If it is not an isolated critical point of  $g(\mu, \cdot)$ , then (i) of Theorem 0.2 obtains. Hence we assume for what follows that  $v = 0$  is an isolated critical point of  $g(\mu, \cdot)$ . This leaves the further possibilities that  $v = 0$  is a (strict) local maximum or minimum for  $g(\mu, \cdot)$  or  $g$  takes on both positive and negative values near 0. We will show the former possibilities result in (ii) of Theorem 0.2 while the latter implies (iii).

Suppose first  $v = 0$  is a strict local minimum for  $g(\mu, \cdot)$ . Let  $c > 0$  and set  $A_{\lambda c} = \{v \in X \mid g(\lambda, v) \leq c\}$ . If  $Q_c$  denotes the component of  $A_{\mu c}$  to which  $v = 0$  belongs, then for  $c$  sufficiently small,  $Q_c$  is a neighborhood of 0 and is strictly interior to the neighborhood of 0 in  $X$  on which  $g(\mu, \cdot)$  is defined. We will show for  $\lambda > \mu$  and near  $\mu$ ,  $g(\lambda, \cdot)$  has at least two distinct critical points in  $Q_c$  other than  $v = 0$ .

Let  $\lambda > \mu$ . Since  $h(z) = o(\|z\|^2)$  at  $z = 0$  and by (1.4),  $\varphi(\lambda, v) = o(\|v\|)$  at  $v = 0$ , we see from (1.5) that the dominating term in  $g(\lambda, \cdot)$  near  $v = 0$  is  $\frac{1}{2}(\mu - \lambda)\|v\|^2$ . Thus for  $\delta = \delta(\lambda)$  sufficiently small,  $(g_v(\lambda, v), v) \leq 0$  for  $0 < \|v\| < \delta$ . Let  $B_r(a)$  denote the open ball in  $X$  of radius  $r$  and center  $a$ . Let  $M = \bar{Q}_c \setminus B_\delta(0)$  and  $\mathcal{O}_{\lambda z} = A_{\lambda z} \cap M$ .

**LEMMA 1.8.** *Let  $g, \lambda, M$  be as above,  $z \in \mathbb{R}$ ,  $K_{\lambda z} = \{v \in M \mid g(\lambda, v) = z, g_v(\lambda, v) = 0\}$ , and  $U$  be any neighborhood of  $K_{\lambda z}$ . Then there exists an  $\eta \in C([0, 1] \times M, M)$  and  $\epsilon > 0$  such that*

$$1^\circ \quad \eta(1, \mathcal{O}_{\lambda, z+\epsilon} \setminus U) \subset \mathcal{O}_{\lambda, z-\epsilon};$$

$$2^\circ \quad \text{if } K_{\lambda z} = \emptyset, \eta(1, \mathcal{O}_{\lambda, z+\epsilon}) \subset \mathcal{O}_{\lambda, z-\epsilon};$$

$$3^\circ \quad \text{if } g(\lambda, \cdot) \text{ is even and } M \text{ is symmetric with respect to the origin, } \eta(t, x) \text{ is odd in } x.$$

*Proof.* The proof of the lemma is standard and we will not carry out the details. (See [11–14]. That is, observing that  $M$  is compact shows the Palais–Smale condition is trivially satisfied. Moreover, for  $\lambda > \mu$  and near  $\mu$ ,  $-g_v(\lambda, v)$

points into  $M$  for  $v \in \partial M$ . Thus minor modifications of the proof of Theorem 4 of [12] or Theorem 1.9 of [13] give the lemma).

Next a closed set  $A \subset M$  is said to have Ljusternik–Schnirelman category one, i.e.,  $\text{cat}_M A = 1$ , if there is a  $\theta \in C([0, 1] \times A, M)$  such that  $\theta(0, v) = v$  and  $\theta(1, v) \equiv \text{constant}$ ; a closed set  $A \subset M$  has Ljusternik–Schnirelman category  $j$ , i.e.,  $\text{cat}_M A = j$ , if there are closed sets  $A_1, \dots, A_j \subset M$  such that  $A \subset \bigcup_j A_i$ ,  $\text{cat}_M A_i = 1$ , and  $j$  is the smallest integer having these properties. For more information on this notion of category, see [11, 14].

With these preliminaries we can define

$$b_i = \inf_{\substack{ACM \\ \text{cat}_M A \geq i}} \max_{v \in A} g(\lambda, v), \quad i = 1, 2. \quad (1.9)$$

From their definition it follows that  $b_1 \leq b_2$ . Since  $\text{cat}_M \partial B_\delta(0) = 2$  and  $g(\lambda, \cdot)$  is negative on  $\partial B_\delta(0)$ ,  $b_2 < 0$ . We claim that  $b_1$  and  $b_2$  are critical values of  $g(\lambda, \cdot)$ . Since  $b_2 < 0$ , they correspond to nonzero critical points. Moreover, if  $b_1 = b_2 = b$ , then  $\text{cat}_M K_{\lambda b} \geq 2$ . Since the proofs of these statements are standard consequences of Lemma 1.8 (see e.g. [11, 14]), we will prove only the last assertion which contains all of the main ideas. (That  $b_1$  is a critical value is easy to see since it is the minimum of  $g(\lambda, \cdot)$  in  $M$  and is not achieved on  $\partial M$ .) Suppose  $b_1 = b_2 = b$  and  $\text{cat}_M K_{\lambda b} < 2$ . Then we can find a neighborhood  $U$  of  $K_{\lambda b}$  such that  $\text{cat}_M U < 2$  [11, 14]. Let  $\epsilon$  be as in Lemma 1.8 with  $z = b$ . By the definition of  $b$ , there is an  $A \subset M$  with  $\text{cat}_M A \geq 2$  and  $\max_{v \in A} g(\lambda, v) \leq b + \epsilon$ . Clearly  $\overline{A \setminus U} \neq \emptyset$  and therefore  $\eta(1, \overline{A \setminus U}) \neq \emptyset$ . Hence  $\max_{v \in \eta(1, \overline{A \setminus U})} g(\lambda, v) \geq b_1$ . But by the choice of  $A$  and 1° of Lemma 1.8,  $\max_{v \in \eta(1, \overline{A \setminus U})} g(\lambda, v) \leq b_1 - \epsilon$ , a contradiction.

Thus for all  $\lambda > \mu$  and near  $\mu$ ,  $g(\lambda, \cdot)$  has at least two distinct nontrivial critical points in  $Q_c$  and alternative (ii) of Theorem 0.2 obtains. If  $g(\lambda, v) < 0$  for  $0 \neq v$  near 0, replacing  $g$  by  $-g$  for  $\lambda < \mu$  gives the same result.

Now we turn to the remaining case where  $g(\mu, \cdot)$  takes on both positive and negative values near  $v = 0$ . We will show (iii) of Theorem 0.2 results here. To begin, let  $B$  be a neighborhood of 0 in  $X$  such that  $g(\lambda, \cdot)$  is defined in  $B$  for all  $\lambda$  near  $\mu$  and  $v = 0$  is the only critical point of  $g(\mu, \cdot)$  in  $B$ . As with the earlier case, a major step in the proof is to find an analog of Lemma 1.8. This is more difficult here since for small  $c$ , a level set of  $g(\mu, \cdot)$ , i.e.,  $\partial Q_c$ , no longer bounds a compact neighborhood of 0 in  $B$ . Nevertheless the level sets for  $g(\mu, \cdot)$  will play a role in constructing a neighborhood of 0 which will be suitable for our purposes.

To begin, note from (1.7) that

$$g_v(\lambda, v) = (\mu - \lambda)v + PH(v + \varphi(\lambda, v)) \quad (1.10)$$

since the right-hand side of (1.10) is  $C^1$ , we see that even though  $\varphi(\lambda, v)$  and  $f(\lambda, v + \varphi(\lambda, v))$  are only  $C^1$  functions of  $v$ ,  $g$  is  $C^2$  in  $v$ .

Let  $\hat{Q}$  be a compact neighborhood of 0 contained in the interior of  $B$  and consider the ordinary differential equation

$$\begin{aligned} d\psi/dt &= -g_v(\mu, \psi), \\ \psi(0, x) &= x. \end{aligned} \quad (11.1)$$

This equation possesses a unique solution  $\psi(t, x)$  for all  $x \in \hat{Q}$ . We will use the flow  $\psi(t, x)$  to construct a special neighborhood of 0 for which we get an analog of Lemma 1.8. As a first step, we have

**LEMMA 1.12.** *Let  $S^+ = \{x \in \hat{Q} \setminus \{0\} \mid \psi(t, x) \subset \hat{Q} \text{ for all } t > 0\}$  and  $S^- = \{x \in \hat{Q} \setminus \{0\} \mid \psi(t, x) \subset \hat{Q} \text{ for all } t < 0\}$ . Then  $S^+$  and  $S^-$  are nonempty.*

*Proof.* Since the proofs are similar, we will only show  $S^+ \neq \emptyset$ . Observe that for  $x \in \hat{Q} \setminus \{0\}$ ,  $g(\mu, \psi(t, x))$  decreases as  $t$  increases since

$$(d/dt)g(\mu, \psi(t, x)) = -\|g_v(\mu, \psi(t, x))\|^2 < 0.$$

Let  $(x_m) \subset \hat{Q} \setminus \{0\}$  such that  $x_m \rightarrow 0$  and  $g(\mu, x_m) > 0$ . If  $S^+$  were empty, the orbit  $\psi(-t, x_m)$  could not remain in  $\hat{Q}$  for all  $t > 0$  since  $g$  has no positive critical values in  $\hat{Q}$ . Let  $t_m > 0$  be the smallest value of  $t$  at which  $y_m = \psi(-t, x_m) \in \partial\hat{Q}$ . Since  $x_m \rightarrow 0$  and  $g_v(\mu, x) = o(\|x\|)$  at  $x = 0$ , it follows from (1.11) that  $t_m \rightarrow \infty$  as  $m \rightarrow \infty$ . A subsequence of  $y_m$  converges to  $y \in \partial\hat{Q}$ . Therefore the orbit  $\psi(t, y) \in \hat{Q} \setminus \{0\}$  for all  $t > 0$  so  $y \in S^+$ . This contradiction implies  $S^+ \neq \emptyset$ .

Slightly abusing notation, now let  $A_{\lambda c} = \{x \in B \mid g(\lambda, x) \leq c\}$ ,  $\hat{A}_{\lambda c} = \{\lambda \in B \mid g(\lambda, x) \geq c\}$ ,  $Y_{\lambda c} = A_{\lambda c} \cap \hat{A}_{\lambda, -c}$ , and for  $W \subset X$ ,  $W_\epsilon = \{x \in B \mid \|x - W\| < \epsilon\}$ .

**LEMMA 1.13.** *Let  $U \subset \hat{Q}$  be an open neighborhood of  $S^-$ . Then there are constants  $c_0, \epsilon_0 > 0$  such that for all  $c \in (0, c_0]$ ,  $\epsilon \in (0, \epsilon_0]$  and  $x \in S_\epsilon^+ \cap Y_{\mu c}$ , the orbit  $\psi(t, x)$  can only exit from  $Y_{\mu c} \cap \hat{Q}$  in  $g(\mu, \cdot)^{-1}(-c) \cap U$ .*

*Proof.* If not, there exist sequences  $c_m \downarrow 0$ ,  $\epsilon_m \downarrow 0$ , and  $x_m \in S_{\epsilon_m}^+ \cap Y_{\mu c_m}$  such that the orbit  $\psi(t, x_m)$  exits from  $Y_{\mu c_m} \cap \hat{Q}$  outside of  $g(\mu, \cdot)^{-1}(-c_m) \cap U$ . Let  $z_m \in \partial\hat{Q} \cup (g(\mu, \cdot)^{-1}(-c_m) \setminus U)$  be the point at which the orbit first exits from  $Y_{\mu c_m}$ . Along a subsequence we have  $z_m \rightarrow z \in g(\mu, \cdot)^{-1}(0) \setminus U$ . Since  $z$  is not a critical point for  $g(\mu, \cdot)$ , the orbit  $\psi(t, z)$  will intersect  $g(\mu, \cdot)^{-1}(c)$  and  $g(\mu, \cdot)^{-1}(-c)$  near  $z$  for all small  $c > 0$  and in particular avoids a neighborhood of 0. By the continuous dependence of  $\psi(t, x)$  on  $x$  (for  $x \neq 0$ ), the same is true of  $\psi(t, z_m)$  along our subsequence for  $m$  large. But  $\psi(t, z_m)$  intersects  $x_m$  and  $x_m \rightarrow 0$  as  $m \rightarrow \infty$ , a contradiction.

**Remark 1.14.** By Lemma 1.13 if  $U = S_\delta^-$ ,  $0 < \epsilon \leq \epsilon_0(\delta)$ ,  $x \in S_\epsilon^+ \setminus S^+$ , and  $0 < g(\mu, x) \leq c_0(\delta)$ , the orbit  $\psi(t, x)$  intersects  $g(\mu, \cdot)^{-1}(-c)$  in  $S_\delta^-$  for

$0 < c \leq c_0(\delta)$ . The same argument shows for any  $S_\rho^+$ ,  $0 < \delta \leq \delta_0(\rho)$ ,  $y \in S_\delta^- \setminus S^-$ , and  $-d_0(\rho) \leq g(\mu, y) < 0$ , the orbit  $\psi(-t, y)$  intersects  $g(\mu, \cdot)^{-1}(d)$  in  $S_\rho^+$  for  $0 < d \leq d_0(\rho)$ .

With the aid of these preliminaries we can construct a neighborhood of 0 suitable for our purposes. By Lemma 1.13 and Remark 1.14, given any  $\epsilon > 0$ , there are positive constants  $c$ ,  $\delta$ , and  $\epsilon_1 \geq \epsilon$  such that if  $x \in S_\epsilon^+ \setminus (S^+ \cup S^-)$ , then for some  $t$ ,  $\tau > 0$ ,  $\psi(t, x) \in S_\delta^-$ ,  $g(\mu, \psi(t, x)) = -c$ ,  $\psi(-\tau, x) \in S_{\epsilon_1}^+$ , and  $g(\mu, \psi(-\tau, x)) = c$ . Set  $T^+(x) = t$ ,  $T^-(x) = \tau$ . If  $x \in S_\epsilon^+ \cap S^+$ , we can still define a  $\tau = T^-(x)$  as above for it and we set  $T^+(x) = \infty$ . Likewise if  $x \in S_\epsilon^+ \cap S^-$ , we define a  $t = T^+(x)$  as above and set  $T^-(x) = \infty$ . Last, if  $x = 0$ , set  $T^\pm(x) = \infty$ .

Let  $Q = \{\psi(t, x) \mid x \in S_\epsilon^+, -T^-(x) < t < T^+(x)\}$ .

LEMMA 1.15.  $Q$  is an open neighborhood of 0.

*Proof.* Since  $0 \in S_\epsilon^+$ , a neighborhood of 0 lies in  $Q$ . If  $0 \neq y \in Q$ , then  $y = \psi(t, x)$  for some  $x \in S_\epsilon^+$  and  $-T^-(x) < t < T^+(x)$ . But then if  $B_r(x) \subset S_\epsilon^+$ ,  $\psi(t, B_r(x))$  is a neighborhood of  $y$  and lies in  $Q$  provided that  $r$  is sufficiently small.

LEMMA 1.16. If  $z \in \partial Q$ , then either (i)  $|g(\mu, z)| = c$  or (ii)  $\psi(t, z) \in \partial Q$  for all  $t$  near 0 and  $|g(\mu, z)| < c$ .

*Proof.* Suppose  $z \in \partial Q$  and  $|g(\mu, z)| \neq c$ . Then  $|g(\mu, z)| < c$ . If for small  $|\theta|$ ,  $\psi(\theta, z) \notin \partial Q$ , then either  $\psi(\theta, z) \in Q$  or  $\psi(\theta, z) \notin \bar{Q}$ . Since  $Q$  as defined is a union of orbit segments, this second alternative is not possible. If  $\psi(\theta, z) \in Q$ , then the definition of  $Q$  then implies  $z = \psi(-\theta, \psi(\theta, z)) \in Q \cup g(\mu, \cdot)^{-1}(c) \cup g(\mu, \cdot)^{-1}(-c)$ , a contradiction.

Now we can complete the proof of Theorem 0.2. We will show for  $\lambda > \mu$  and  $\lambda < \mu$ ,  $g(\lambda, \cdot)$  possesses a nonzero critical point in  $Q$ . The proof is related to one given in [15] where a simpler situation is treated. Suppose first that  $\lambda > \mu$ . As with case (ii), the dominating term in  $g(\lambda, \cdot)$  near  $v = 0$  is  $\frac{1}{2}(\mu - \lambda) \|v\|^2 \geq 0$  and there is an  $r_0 > 0$  such that for  $r \in (0, r_0]$  and  $v \in B_{r_0}(0)$ ,

$$g(\lambda, v) \geq \frac{1}{4}(\mu - \lambda) \|v\|^2. \quad (1.17)$$

Let  $\Gamma = \{\theta \in C([0, 1], \bar{Q}) \mid \theta(0) = 0, g(\mu, \theta(1)) = -c\}$ . Clearly  $\Gamma \neq \emptyset$ . Define

$$b = \inf_{\theta \in \Gamma} \max_{t \in [0, 1]} g(\lambda, \theta(t)). \quad (1.18)$$

We can assume  $g(\lambda, x) \leq 0$  for  $x \in g(\mu, \cdot)^{-1}(-c)$ . Since each curve  $\theta \in \Gamma$  joins 0 and  $g(\mu, \cdot)^{-1}(-c) \cap \partial Q$ , it must cross  $\partial B_{r_0}$ . Therefore  $\max_{t \in [0, 1]} g(\lambda, \theta(t)) \geq \kappa = \frac{1}{4}(\mu - \lambda) r_0^2 > 0$ . Therefore  $0 < \kappa \leq b < \infty$ . We claim there exists  $v \in Q$  such that  $g(\lambda, v) = b$  and  $g_v(\lambda, v) = 0$ , i.e.,  $b$  is a critical value of  $g(\lambda, \cdot)$  in  $Q$ . To verify our claim, we require the following lemma. Slightly abusing notation, let  $\mathcal{O}_{\lambda z} = A_{\lambda z} \cap \bar{Q}$  and  $K_{\lambda z} = \{x \in \bar{Q} \mid g(\lambda, x) = z, g_v(\lambda, x) = 0\}$ .

LEMMA 1.19. *If  $Q$  is as above,  $z > -c$ ,  $\delta_1 > 0$ , and  $U$  any neighborhood of  $K_{\lambda z}$ , then there exists an  $\eta \in C([0, 1] \times \bar{Q}, \bar{Q})$  and  $\delta > 0$  such that:*

$$1^\circ \quad \eta(t, v) = v \text{ if } v \notin g(\lambda, \cdot)^{-1}[z - \delta_1, z + \delta_1], t \in [0, 1];$$

$$2^\circ \quad \eta(1, \mathcal{O}_{\lambda, z+\delta} \setminus U) \subset \mathcal{O}_{\lambda, z-\delta};$$

$$3^\circ \quad \text{if } K_{\lambda z} = \emptyset, \eta(1, \mathcal{O}_{\lambda, z+\delta}) \subset \mathcal{O}_{\lambda, z-\delta};$$

4° *if  $g(\lambda, \cdot)$  is even and  $Q$  is symmetric with respect to the origin, then  $\eta(t, v)$  is odd in  $v$ .*

*Proof.* As with Lemma 1.8, the proof involves only small modifications of that of Theorem 4 of [12] or Theorem 1.9 of [13]. We will indicate the changes required here and refer the reader to [12] or [13] for details. Note in particular that  $\bar{Q}$  is compact so that the Palais–Smale condition is trivially satisfied. Moreover,  $g_v(\mu, v)|_{\partial Q}$  extends to a pseudogradient vector field  $\hat{V}$  for  $g_v(\lambda, \cdot)$  in  $\bar{Q}$  [11, 13]. The function  $\eta$  is then defined by an ordinary differential equation

$$d\eta/dt = -\chi(\eta) \hat{V}(\eta), \quad \eta(0, x) = x \quad (1.20)$$

where among other things  $\chi(x)$  is a scalar locally Lipschitz continuous function and  $\chi(x) = 0$  when  $g(\mu, x) = -c$ . The behavior of  $\hat{V}$  on  $\partial Q \cap g(\mu, \cdot)^{-1}(-c, c)$  implies this set is invariant under the flow defined by (1.20).

Since  $-\hat{V}$  points into  $Q$  on  $\partial Q \cap g(\mu, \cdot)^{-1}(c)$ ,  $\eta(t, \cdot)$  maps that set into  $Q$ . Hence  $\eta(t, \cdot): Q \rightarrow Q$ . For the remaining assertions see [12] or [13].

To show that  $b$  is a critical value of  $g(\lambda, \cdot)$ , assume the contrary. Then with  $0 < \delta_1 < b$ ,  $x = b$  and  $\delta$  as in Lemma 1.19, by the definition of  $b$ , there is a  $\theta \in \Gamma$  such that

$$\max_{t \in [0, 1]} g(\lambda, \theta(t)) \leq b + \delta.$$

By 3° of Lemma 1.19,

$$\max_{t \in [0, 1]} g(\lambda, \eta(1, \theta(t))) \leq b - \delta. \quad (1.21)$$

But  $\eta(1, \theta(t)) \in C([0, 1], \bar{Q})$  and by 1° of Lemma 1.19  $\eta(1, 0) = 0$  and  $g(\mu, \theta(1)) = -c$ . Hence  $\eta(1, \theta(t)) \in \Gamma$  contrary to (1.12).

Last, if  $\lambda > \mu$ , we get the result on replacing  $g$  by  $-g$ .

We will conclude this section with a few remarks.

**Remark 1.22.** Suppose that  $g(\mu, \cdot)^{-1}(-c)$  contains more than one component. The argument given above shows for each such component  $\xi$  we can define a class  $T_\xi = \{\theta \in C([0, 1], Q) \mid \theta(0) = 0, \theta(1) \in \xi\}$  and obtain a critical value of  $g(\lambda, \cdot)$  as in (1.18) with  $\Gamma$  replaced by  $\Gamma_\xi$ . We do not know whether one obtains more than one critical point of  $g(\lambda, \cdot)$  in this fashion.

**Remark 1.23.** Actually once  $Q$  has been constructed, and  $\hat{V}$  defined, another



proof of the existence of a critical point of  $g(\lambda, \cdot)$  can be given without characterizing the corresponding critical value. Namely observe that if the flow defined by (1.20) (with  $\chi \equiv 1$ ) is reversed and there are no critical points in  $Q \setminus B_{r_0}(0)$ , the orbits passing through all points on  $\partial B_{r_0}(0)$  and on  $\partial Q \cap g(\mu, \cdot)^{-1}(-c)$  must exit from  $\bar{Q}$  on  $T = \partial Q \cap g(\mu, \cdot)^{-1}(c)$  and every point on  $T$  lies on such an orbit. But this is not possible.

*Remark 1.24.* The proof of Theorem 0.2 shows that the structure of the solution set of (0.1) near  $(\lambda, u) = (\mu, 0)$  is determined by the behavior of  $g(\mu, v)$  near  $v = 0$ . This in turn can be obtained from that of  $h$  in some interesting cases. For example, if  $\mu$  is the smallest (resp. largest) eigenvalue of  $L$ ,  $g \geq h$  (resp.  $g \leq h$ ) so if  $h \geq 0$  (resp.  $h \leq 0$ ), we are in case (i) or (ii). As another example, suppose  $H(z) = H_{k-1}(z) + o(\|z\|^{k-1})$  at  $z = 0$  where  $H_{k-1}$  is homogeneous of degree  $k-1 > 1$ . Then  $h(z) = h_k(z) + o(\|z\|^k)$  at  $z = 0$  with  $h_k$  homogeneous of degree  $k$  and  $h'_k = H_{k-1}$ . If further  $h_k(z) \neq 0$  for all  $z \in X \setminus \{0\}$  (and therefore  $k$  is even),

$$|h_k(z)| \geq c \|z\|^k \quad \text{and} \quad \|H_{k-1}(z)\| \geq ck \|z\|^{k-1},$$

for some constant  $c$ . But (1.3) and (1.4) and the above then imply

$$\|\varphi(\mu, z)\| \leq c_1 \|z\|^{k-1}$$

so since  $k > 2$  from (1.5) we see that

$$|g(\lambda, v)| \geq \text{const} \|v\|^k$$

and we are in case (i) or (ii) again.

*Remark 1.25.* In [3, 4], a more general term than  $u$  was permitted on the right-hand side of (0.1). That can be done here also and a more general  $\lambda$  dependence permitted provided the basic qualitative features of  $g(\lambda, u)$  we exploited do not change.

## 2. THE SYMMETRIC CASE

When  $f$  is even, the set of solutions of (0.1) possesses a richer structure. This case has been studied by Böhme, Marino, and Clark as mentioned in the Introduction. To describe our results, we will use the notion of genus. If  $E$  is a real Banach space and  $A \subset E \setminus \{0\}$  is closed and symmetric with respect to the origin, we say  $A$  has *genus*  $k$ , denoted by  $\gamma(A) = k$ , if there exists  $\varphi \in C(A, \mathbb{R}^k \setminus \{0\})$  with  $\varphi$  odd and  $k$  is the smallest integer having these properties. The properties of genus we require are stated in the following lemma. For a proof see e.g. [16] or [17].

LEMMA 2.1. *If  $E$  is a real Banach space and  $A, B \subset E \setminus \{0\}$  are closed and symmetric with respect to the origin, then*

1° *if there is an odd  $\chi \in C(A, B)$ , then  $\gamma(A) \leq \gamma(B)$  with equality if  $\chi$  is a homeomorphism;*

2° *if  $A \subset B$ ,  $\gamma(A) \leq \gamma(B)$ ;*

3°  *$\gamma(A \cup B) \leq \gamma(A) + \gamma(B)$ ;*

4° *if  $\gamma(B) < \infty$ ,  $\gamma(\overline{AB}) \geq \gamma(A) - \gamma(B)$ ;*

5° *if  $A$  is compact and  $\gamma(A) < \infty$ , there is a  $\delta > 0$  such that  $\gamma(N_\delta(A)) = \gamma(A)$  where  $N_\delta(A) = \{x \in E \mid \|x - A\| \leq \delta\}$ ;*

6° *if  $E = \mathbb{R}^n$  and  $A$  is the boundary of a neighborhood of 0, then  $\gamma(A) = n$ .*

Let  $f$  be as in Section 1 and let  $f$  be even. Then the function  $g(\lambda, v)$  as defined earlier is even in  $v$ , the pseudogradient vector field introduced in Section 1 can be made odd, and the sets  $M$  and  $Q$  constructed to prove (ii) and (iii) of Theorem 0.2 can be assumed to be symmetric with respect to the origin. Since we do not feel we have the appropriate symmetric analog of Theorem 0.2, we will state our results as two separate theorems.

THEOREM 2.2. *Let the hypotheses of Theorem 0.2 be satisfied and  $f$  be even. If  $v = 0$  is an isolated critical point and local minimum (maximum) for  $g(\mu, \cdot)$  then there is a right (left) neighborhood  $\Lambda$  of  $\mu$  such that for all  $\lambda \in \Lambda \setminus \{\mu\}$ , (0.1) possesses at least  $n$  distinct pairs of nontrivial solutions.*

*Proof.* Let  $M$  be as in Section 1. We will find the required number of critical points of  $g(\lambda, \cdot)$  in  $M$ . Define

$$b_i = \inf_{\substack{A \subset M \\ \gamma(A) \geq i}} \max_{v \in A} g(\lambda, v), \quad 1 \leq i \leq n. \quad (2.3)$$

Since  $\dim X = n$ , by 2° and 6° of Lemma 2.1,  $\gamma(\partial B_{\delta(0)}) = n \leq \gamma(M) (= n)$ . Hence the numbers  $b_i$ ,  $1 \leq i \leq n$ , are well defined. It is clear that  $b_1 \leq \dots \leq b_n$  and as in Section 1,  $b_n < 0$ . We claim that these numbers are critical values of  $g(\lambda, \cdot)$  (which correspond to nontrivial critical points of  $g(\lambda, \cdot)$  by the above remarks) and if  $b_{i+1} = \dots = b_{i+p} \equiv b$ ,  $\gamma(K_{\lambda b}) \geq p$ . The theorem then follows.

It suffices to prove the last statement. Suppose  $\gamma(K_{\lambda b}) < p$ . Then by 5° of Lemma 2.1, there is a  $\delta > 0$  such that  $\gamma(N_\delta(K_{\lambda b})) \leq p - 1$ . By the construction of  $M$ , we can assume  $N_\delta(K_{\lambda b}) \subset \text{int } M$  ( $\text{int} \equiv \text{interior}$ ). Let  $\epsilon$  be as in Lemma 1.8 with  $U = \text{int } N_\delta(K_{\lambda b})$ . By the definition of  $b$ , there is an  $A \subset M$  with  $\gamma(A) \geq i + p$  and  $\max_{v \in A} g(\lambda, v) \leq b + \epsilon$ . By 3° of Lemma 1.8 and 1°, 4° of Lemma 2.1,  $\gamma(\eta(1, A \setminus U)) \geq \gamma(A \setminus U) \geq \gamma(A) - \gamma(\overline{U}) \geq i + p - (p - 1) = i + 1$ . Hence  $\eta(1, A \setminus U)$  is admissible for the computation of  $b_{i+1}$  and  $b \leq \max_{v \in \eta(1, A \setminus U)} g(\lambda, v)$ . But this is impossible since  $\eta(1, \mathcal{U}_{\lambda, b-\epsilon} \setminus U) \subset \mathcal{U}_{\lambda, b-\epsilon}$  and the theorem is proved.

Next we treat the symmetric analog of (iii) of Theorem 0.2. The proof is related to Clark's [10] for his case.

**THEOREM 2.4.** *Let the hypotheses of Theorem 0.2 be satisfied and  $f$  be even. Suppose  $v = 0$  is an isolated critical point for  $g(\mu, \cdot)$  and  $g(\mu, \cdot)$  takes on both positive and negative values near 0. If  $\gamma(S^- \cap \partial Q) = k$ , there is a right neighborhood,  $\mathcal{J}^+$ , of  $\mu$  such that for all  $\lambda \in \mathcal{J}^+ \setminus \{\mu\}$ , (0.1) possesses at least  $n - k$  distinct pairs of nontrivial solutions; if  $\gamma(S^+ \cap \partial Q) = m$ , there is a left neighborhood,  $\mathcal{J}^-$ , of  $\mu$  such that for all  $\lambda \in \mathcal{J}^- \setminus \{\mu\}$ , (0.1) possesses at least  $n - m$  distinct pairs of nontrivial solutions.*

*Proof.* Suppose first that  $0 > \mu - \lambda$  is small. Let  $T = S^- \cap \partial Q$  and  $z = \max_{v \in T} g(\lambda, v)$ . Then  $\mathcal{O}_{\lambda z} \supset T$  so by 2° of Lemma 2.1,  $\gamma(\mathcal{O}_{\lambda z}) \geq \gamma(T)$ . For  $\lambda$  sufficiently near  $\mu$ , we can assume (via 2° and 5° of Lemma 2.1) that  $\gamma(\mathcal{O}_{\lambda z}) = \gamma(T) = k$ . For  $z \leq r < 0$ , consider  $\gamma(\mathcal{O}_{\lambda r})$ . By 2° of Lemma 2.1 again,  $\gamma(\mathcal{O}_{\lambda r})$  is a monotone nondecreasing function of  $r$ . For  $r$  near  $z$ ,  $\gamma(\mathcal{O}_{\lambda r}) = k$ ; for  $r$  near 0, (1.5) shows  $\mathcal{O}_{\lambda r} \supset \partial B_\rho(0)$  for appropriate  $\rho$  near 0 so by 6° of Lemma 2.1,  $\gamma(\mathcal{O}_{\lambda r}) = n$ . Since  $\gamma(\mathcal{O}_{\lambda r})$  is integer valued, it has at most  $n - k$  discontinuities in  $(z, 0)$ . Thus the first assertion of Theorem 2.4 follows once we show

**LEMMA 2.5.** *If  $\gamma(\mathcal{O}_{\lambda r})$  has a discontinuity at  $s \in (z, 0)$  and  $p = \lim_{\delta \downarrow 0} [\gamma(\mathcal{O}_{\lambda, s+\delta}) - \gamma(\mathcal{O}_{\lambda, s-\delta})]$ , then  $\gamma(K_{\lambda, s}) \geq p$ .*

*Proof.* We use a familiar argument. If  $\gamma(K_{\lambda p}) \leq p - 1$ , then there is an  $a > 0$  so that  $\gamma(N_a(K_{\lambda s})) \leq p - 1$ . By Lemma 1.19,  $\eta(1, \mathcal{O}_{\lambda, s+\epsilon} \setminus N_a(K_{\lambda s})) \subset \mathcal{O}_{\lambda, s-\epsilon}$ . Therefore  $\gamma(\mathcal{O}_{\lambda, s-\epsilon}) \geq \gamma(\eta(1, \mathcal{O}_{\lambda, s+\epsilon} \setminus N_a(K_{\lambda s}))) \geq \gamma(\mathcal{O}_{\lambda, s+\epsilon} \setminus N_a(K_{\lambda s})) \geq \gamma(\mathcal{O}_{\lambda, s+\epsilon}) - (p - 1)$ . But this is contrary to the definition of  $p$ .

Finally to obtain the second assertion of Theorem 2.4, let  $\lambda < \mu$  and replace  $g(\lambda, v)$  by  $-g(\lambda, v)$ . The result then follows from the case just treated.

**Remark 2.6.** We suspect that for each  $\lambda \in \mathcal{J}^- \setminus \{\mu\}$ , (0.1) possesses at least  $k$  distinct pairs of nontrivial solutions and for each  $\lambda \in \mathcal{J}^+ \setminus \{\mu\}$ , (0.1) has at least  $m$  distinct pairs of nontrivial solutions. This would follow from Theorem 2.4 if we could show  $m + k = n$ . (See also [10] where Clark conjectured a related result.) However, we can only prove the weaker statement

**LEMMA 2.7.**  $m + k \geq n$ .

*Proof.* If in the construction of  $Q$ , we replace  $c$  by  $d \in (0, c)$ , we obtain a neighborhood of 0,  $Q_d \subset Q \equiv Q_c$  having the same properties as  $Q$ . If  $z \in \partial Q_d \cap g(\mu, \cdot)^{-1}(-d)$ , then there is a  $t(z) > 0$  such that  $g(\mu, \psi(t(z), z)) = -c$ . The map  $\Psi(z) = \psi(t(z), z)$  continuously maps  $\partial Q_d \cap g(\mu, \cdot)^{-1}(-d)$  into  $\partial Q_c \cap g(\mu, \cdot)^{-1}(-c)$  and  $\Psi(S^- \cap \partial Q_d) = S^- \cap \partial Q_c$ . Since  $\gamma(S^- \cap \partial Q_c) = k$ , by 4° and 2° of Lemma 2.1, there is a  $\delta > 0$  such that  $\gamma(N_\delta(S^-) \cap \partial Q_c) = k$ .

For  $d$  sufficiently small, we can assume  $\Psi(\partial Q_d \cap g(\mu, \cdot)^{-1}(-d)) \subset N_\delta(S^-) \cap \partial Q_c$ . Therefore by 1° and 2° of Lemma 2.1,  $\gamma(\partial Q_d \cap g(\mu, \cdot)^{-1}(-d)) = k$ . Similarly we can assume  $\gamma(\partial Q_d \cap g(\mu, \cdot)^{-1}(d)) = m$ . If  $z \in \partial Q_d \setminus g(\mu, \cdot)^{-1}(-d)$ , there is a  $\tau(z) \leq 0$  such that  $g(\mu, \psi(\tau(z), z)) = d$ , i.e., there is a retraction of  $\overline{\partial Q_d \setminus g(\mu, \cdot)^{-1}(-d)}$  onto  $\partial Q_d \cap g(\mu, \cdot)^{-1}(d)$ . Hence  $\gamma(\overline{\partial Q_d \setminus g(\mu, \cdot)^{-1}(-d)}) \leq \gamma(\partial Q_d \cap g(\mu, \cdot)^{-1}(d)) = m \leq \gamma(\overline{\partial Q_d \setminus g(\mu, \cdot)^{-1}(-d)})$ . Thus we have equality in this equation. Since  $\gamma(\partial Q_d) = n$ , by 3° of Lemma 2.1,  $n = \gamma(\overline{\partial Q_d \setminus g(\mu, \cdot)^{-1}(-d)} \cup (\partial Q_d \cap g(\mu, \cdot)^{-1}(-d))) \leq \gamma(\overline{\partial Q_d \setminus g(\mu, \cdot)^{-1}(-d)}) + \gamma(\partial Q_d \cap g(\mu, \cdot)^{-1}(-d)) = m + k$ .

To further support Remark 2.6, we give a minimax characterization of critical values of  $g(\lambda, \cdot)$  which is more in the spirit of the proof of (iii) of Theorem 0.2. Let  $\lambda < \mu$ . For  $1 \leq j \leq k$  and  $K \subset \partial Q \cap S^-$  with  $\gamma(K) = j$ , define  $\Phi(K) = \{\psi(t, x) \mid t \in (-\infty, 0], x \in K\}$ , i.e., we use  $\psi$  to cone  $K$  over 0. Let  $\mathcal{F} = \{\chi \in C(\bar{Q}, \bar{Q}) \mid \chi \text{ is odd, one to one, and } \chi(v) = v \text{ for } v \in \partial Q \cap S^-\}$ . Set  $\Gamma_j = \{\chi(\Phi(K)) \mid \chi \in \mathcal{F} \text{ and } K \in \partial Q \cap S^- \text{ with } \gamma(K) \geq j\}$  for  $1 \leq j \leq k$ . Note that if  $\chi \in \mathcal{F}$  and  $A \in \Gamma_j$ , then  $\chi(A) \in \Gamma_j$ . Finally define

$$b_i = \inf_{A \in \Gamma_i} \max_{v \in A} g(\lambda, v), \quad 1 \leq j \leq k. \quad (2.8)$$

Clearly  $b_1 \leq \dots \leq b_k$  and for  $\lambda$  near  $\mu$ ,  $b_1 > 0$  as in (iii) of Theorem 0.2.

**THEOREM 2.9.**  $b_j$  is a critical value of  $g(\lambda, \cdot)$ ,  $1 \leq j \leq k$ .

*Proof.* Since  $\eta(1, \cdot): \Gamma_j \rightarrow \Gamma_j$ , this follows by a familiar argument.

A similar construction and theorem can be given for  $\lambda > \mu$ . Unfortunately we are unable to prove a "multiplicity theorem" saying if  $b_{i+1} = \dots = b_{i+p} \equiv b$ , then  $\gamma(K_{\lambda k}) \geq p$ . If this were true, our conjecture of Remark 2.6 would be verified.

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